ELSEVIER
Topology and its Applications 119 (2002) 219–227
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Strongly Hewitt spaces

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Received 19 May 2000; received in revised form 21 September 2000

Abstract

We investigate completely regular spaces X such that for any sequence (x_n) in $\beta X \setminus X$ there exists a real continuous function on βX which is positive on X and vanishes on some subsequence of (x_n). Applications to spaces of continuous functions $C(X,E)$ are included.

AMS classification: 46A30; 54C35

Keywords: Baire spaces; Baire-like spaces; Hewitt spaces; Bornological spaces

1. Introduction and preliminary facts

Throughout this note “lcs” will stand for “Hausdorff locally convex topological vector space”. The word “space” will mean “completely regular Hausdorff topological space”. By $\mathbb{R}$ and $\mathbb{N}$ we denote the sets of the real and natural numbers, respectively. Denote by X^* the Stone–Čech remainder $(\beta X \setminus X)$ of X and let $C(X)$ (respectively $C^*(X)$) be the space of continuous (respectively continuous and bounded) real-valued functions on X. $C_c(X)$ and $C_p(X)$ denote the space $C(X)$ endowed with the compact-open topology and the topology of the pointwise convergence, respectively. Recall, that a subspace S of X is C^**-embedded* in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$. A space X is called a *Hewitt space* (or realcompact space), cf. [9], if for every point x in X^* there exists $f \in C(\beta X)$ which is positive on X and $f(x) = 0$. For a space X by βX and υX we denote the Stone–Čech and the Hewitt compactification of X, respectively. Note that X is a Hewitt space iff $X = \upsilon X$. We shall say that X is a μ*-space* if every bounding

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PII: S0166-8641(01)00063-3

subset of X is relatively compact. Recall that $A \subset X$ is *bounding* if $f(A)$ is bounded for every $f \in C(X)$. It is known that A is bounding iff $\overline{A}^{\beta X} \subset \upsilon X$, cf. [16, 10.1.17].

Every Hewitt space is a μ-space; the converse does not hold. For another characterizations of Hewitt spaces see [9,20]. Recall also that according to the Nachbin–Shirota theorem, cf. [16, 10.1.12], X is a Hewitt space iff the space $C_c(X)$ is *bornological*, i.e., every linear map from $C_c(X)$ into a lcs which transforms bounded sets into bounded sets is continuous.

In this note we investigate spaces X (under the name *strongly Hewitt*) such that (H) *for every sequence* (x_n) *in* X^* *there exists* $f \in C(\beta X)$ *which is positive on* X *and vanishes on some subsequence of* (x_n).

It is known, cf. [20, Exer. 1B. 4], that if X is locally compact σ-compact, then X^* is a zero set in βX, so X has property (H). Clearly every strongly Hewitt space is Hewitt. The main result of this note says that X is strongly Hewitt iff X is Hewitt and X^* is countably compact and every strongly Hewitt space of pointwise countable type is locally compact, Theorem 1. Hence every locally compact Hewitt space is strongly Hewitt but there exist strongly Hewitt spaces which are not locally compact, Example A. We apply our results to show that the space $C_c(X, E)$ of continuous functions defined on a locally compact space X into a metrizable locally convex space E is Baire-like and bornological iff E is barrelled and X is Hewitt, Theorem 2. Hence $C_c(X)$ is Baire-like for every strongly Hewitt space X. Nevertheless, we give an example of a locally compact and Hewitt space X such that $C_c(X)$ is not Baire, Example C. Also we discuss the problem if the strongly Hewitt property is inherited by the ℓ-equivalence (in the sense of Arhangel'skii), Theorem 3.

We show that if X is strongly Hewitt, then every infinite subset P of X^* contains an infinite subset which is relatively compact in X^* and C^*-embedded in βX, Proposition 1.

On the other hand, in [4, Example 1.11], Baumgartner and van Douwen constructed a separable first countable locally compact Hewitt space X (hence strongly Hewitt) for which X^* contains a discrete countable subset which is not C^*-embedded in βX. This result combined with [4, 1.2] applies also to provide an example of a locally compact Hewitt space X such that X^* contains a sequence (x_n) for which does not exist $f \in C(\beta X)$ which is positive on X and vanishes on (x_n). By Gillman and Henriksen [10, 2.7], every σ-compact subspace of X^* is C^*-embedded in βX provided X is locally compact and σ-compact.

The space $\mathbb{Q}$ of the rational numbers is not strongly Hewitt but using [5, pp. 8–9], the remainder $\mathbb{Q}^*$ is a $\beta\omega$-space, i.e., if D is a countable discrete subset of $\mathbb{Q}^*$ and $\overline{D}$ (the closure in $\mathbb{Q}^*$) is compact, then $\overline{D} = \beta D$, so D is C^*-embedded in $\beta\mathbb{Q}$. Recall, as it follows from [9, 14.M.2], that the space $\mathbb{Q}^*$ contains a countable subset which is not C^*-embedded in $\beta\mathbb{Q}$.

2. Results

Recall that X is an *F-space* if its every cozero-set is C^*-embedded in X. By [9, 14.N.5], it is known that in every F-space X any countable subset of X is C^*-embedded in X.

Another source of spaces with a similar property is given by $\beta\omega$-spaces. For strongly Hewitt spaces we note the following fact; its proof uses some idea from Negrepontis' proof, cf. [15, 3.2], of the Gillman–Henriksen result of [10, 2.7].

Proposition 1. *If X is strongly Hewitt, then every infinite subset D of X^* contains an infinite subset S which is relatively compact in X^* and C^*-embedded in βX.*

Proof. Assume that (x_n) is a sequence of different elements in D and let $f:\beta X \to [0,1]$ be a continuous function which is positive on X and vanishes on a subsequence of (x_n).

Let

$$S = \{x_n\colon n \in \mathbb{N}\} \cap f^{-1}\{0\}, \quad Y_n = \left\{x \in \beta X\colon |f(x)| \geqslant n^{-1}\right\}, \quad n \in \mathbb{N},$$

$$X_1 = S \cup \bigcup_n Y_n.$$

The space X_1 is regular and σ-compact, so it is normal. Since S is closed in X_1, it is C^*-embedded in X_1; hence it is C^*-embedded in βX_1. Since $X \subset X_1 \subset \beta X$, the equality $\beta X_1 = \beta X$ holds. □

The following fact extends [9, 9.12].

Corollary 1. *Let X be a strongly Hewitt space. Then every infinite closed set in X^* contains a copy of $\beta\mathbb{N}$.*

On the other hand, applying [4, 1.2 and 1.10] one gets the following:

Proposition 2. *Let X be a metrizable and locally compact space without isolated points. If every countable subset of X^* is C^*-embedded in βX, then for any countable set D in X^* there exists $f \in C(\beta X)$ which is positive on X and vanishes on D.*

Recall, cf. [1], that a space X is of *pointwise countable type* if each $x \in X$ is contained in a compact set $K \subset X$ of countable character in X. The spaces of pointwise countable type include, in particular, all spaces satisfying the first axiom of countability and all Čech-complete spaces.

The main result of this note is the following:

Theorem 1.

(a) *A space X is strongly Hewitt iff it is Hewitt and X^* is countably compact. In particular every locally compact Hewitt space is strongly Hewitt.*

(b) *Every strongly Hewitt space of pointwise countable type is locally compact.*

Proof. (a) Assume that X is strongly Hewitt. Let $P \subset X^*$ be an arbitrary infinite set and let (x_n) be a sequence of different elements of P. By assumption there exists a continuous function $f:\beta X\to[0,1]$ which is positive on X and vanishes on some subsequence (x_{kn}) of (x_n). Then

$$\{x_{k_n}\colon n\in\mathbb{N}\}\subset f^{-1}(0)\subset X^*,$$

so $\{x_{k_n}\colon n\in\mathbb{N}\}^d\subset f^{-1}(0)$ and clearly $\{x_{k_n}\colon n\in\mathbb{N}\}^d$ is non-empty, where A^d denotes the set of all accumulation points of A. This proves that P has an accumulation point.

For the converse assume that X is Hewitt and every infinite subset of X^* has an accumulation point in X^*. Let (x_n) be a sequence in X^*. If the set $P=\{x_n\colon n\in\mathbb{N}\}$ is finite, then by the Hewitt property of X there exists a continuous function $f:\beta X\to[0,1]$ which is positive on X and vanishes on a subsequence of (x_n). Now assume that $P=\{x_n\colon n\in\mathbb{N}\}$ is infinite. Let $p\in P^d\setminus X$. Then there exists a continuous function $f:\beta X\to[0,1]$ which is positive on X and vanishes on p. Note that for every $r>0$ the set $P\cap f^{-1}([0,r))$ is infinite, since $f^{-1}([0,r))$ is a neighbourhood of the point $p\in P^d$. Two cases are possible:

Case 1. The set $P\cap f^{-1}(0)$ is infinite. Then f is positive on X and vanishes on some subsequence of the sequence (x_n).

Case 2. The set $P\cap f^{-1}(0)$ is finite. Since for every $r>0$ the set $P\cap f^{-1}([0,r))$ is infinite, there exists a sequence (p_n) of different elements of the set P such that the sequence $(f(p_n))$ is strictly decreasing and converges to zero. Put $P_0=\{p_n\colon n\in\mathbb{N}\}$. Let $s_0=1$ and let $s_k\in(f(p_{k+1}),f(p_k))$ for all $k\in\mathbb{N}$. Then (s_k) is decreasing and converges to zero. Put $F_k=f^{-1}([s_k,s_{k-1}])$ for $k\in\mathbb{N}$. Clearly the sets F_k are compact and $p_k\in F_n$ iff $k=n$.

Moreover,

$$X\subset f^{-1}\bigl((0,1]\bigr)=\bigcup_k F_k$$

and

$$P_0\cap F_k=\{p_k\},\quad k\in\mathbb{N}.$$

If $f(x)=c>0$, then $x\in f^{-1}((2^{-1}c,1])$. Since $f(p_k)\to 0$, we conclude that $x\notin P_0^d$. Hence, if $x\in P_0^d$, then $x\in f^{-1}(0)$, so $x\notin\bigcup_k F_k$. This proves that $P_0^d\cap(\bigcup_k F_k)=\emptyset$. Since X is Hewitt, for every $k\in\mathbb{N}$ there exists a continuous function $f_k:\beta X\to[0,1]$ which is positive on X and vanishes on p_k.

Put

$$T_n^k=f_k^{-1}\bigl([n^{-1},1]\bigr),\quad n,k\in\mathbb{N}.$$

Then $X\subset f_k^{-1}((0,1])=\bigcup_n T_n^k$ and $p_k\notin T_n^k$ for all $k,n\in\mathbb{N}$.

Also

$$X\subset\bigcup_k F_k\cap X\subset\bigcup_k\bigcup_n F_k\cap T_n^k,$$

$$P_0\cap\bigl(F_k\cap T_n^k\bigr)\subset P_0\cap F_k=\{p_k\}.$$

But $p_k\notin F_k\cap T_n^k$, so $P_0\cap(F_k\cap T_n^k)=\emptyset$ for all $n,k\in\mathbb{N}$. Hence $P_0\cap W=\emptyset$ and $P_0^d\cap W\subset P_0^d\cap\bigcup_k F_k=\emptyset$, where $W=\bigcup_k\bigcup_n F_k\cap T_n^k$.

Therefore

$$\overline{P_0}\cap W=\emptyset.$$

This proves that there exists an infinite subset P_0 of P and an infinite sequence of compact sets (K_n) such that

$$X \subset \bigcup_n K_n \subset \beta X, \qquad \left(\bigcup_n K_n\right) \cap \overline{P_0} = \emptyset.$$

For every $n \in \mathbb{N}$ let $g_n : \beta X \to [0.1]$ be a continuous function such that

$$g_n | K_n = 1, \qquad g_n | \overline{P_0} = 0.$$

Put $g = \sum_n 2^{-n} g_n$. Then the function $g : \beta X \to [0,1]$ is continuous, positive on X and vanishes on some subsequence of (x_n). Thus we proved that for every sequence (x_n) in X^* there exists a continuous function on βX which is positive on X and vanishes on some subsequence of (x_n).

(b) Assume that X is strongly Hewitt of pointwise countable type but is not locally compact. Then there exist $x_0 \in X$ which does not admit a relatively compact neighbourhood, a compact set K in X containing x_0 and a countable (decreasing) basis (U_n) of neighbourhoods of K. For every $n \in \mathbb{N}$ let $x_n \in (\overline{U_n} \setminus X)$, where the closure is taken in βX. Observe that $(\beta X \setminus K) \cap \{x_n\}^d = \emptyset$. Indeed, let $x \in (\beta X \setminus K)$. Let $V \subset \beta X$ be an open neighbourhood of K such that $x \in (\beta X \setminus \overline{V})$. Then there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset V \cap X$, so $\overline{U_{n_0}} \subset \overline{V}$. Since $\{x_n\}^d \subset \overline{U_{n_0}} \subset \overline{V}$, then $x \in \beta X \setminus \{x_n\}^d$. Hence $\{x_n\}^d \subset K$. Clearly $\{x_n\}^d$ is non-empty. This contradicts that X is strongly Hewitt. □

Example A. *There exists a strongly Hewitt space which is not locally compact*. Let P be a countably and non-empty subset of $\mathbb{N}^*$. The subspace $X = \mathbb{N} \cup P$ of $\beta\mathbb{N}$ is Lindelöf and so it is Hewitt. Since every countably and closed subset of $\beta\mathbb{N}$ is finite, cf. [20, p. 71], the space $X^* = \mathbb{N}^* \setminus P$ is countably compact and X is not locally compact. Clearly, by Theorem 1(a), X is strongly Hewitt.

From Theorem 1 it follows directly the following

Example B. *The space $\mathbb{R}^{\mathbb{N}}$ is a Hewitt space which is not strongly Hewitt.*

3. Applications to continuous function spaces

It is well known that if X is either a complete metric space or a locally compact Hausdorff space, then the intersection of countably many dense and open subsets of X is a dense subset of X. Spaces with the property as above are called *Baire spaces*. A new line of research concerning Baire-type conditions started with the Amemiya–Komura property, cf. [16, 8.2.12], and so, Saxon [17] defined a lcs E to be *Baire-like* if given an increasing sequence (A_n) of closed absolutely convex subsets of E covering E, there is an integer $n \in \mathbb{N}$ such that A_n is a neighbourhood of zero. When the sequence (A_n) is required to be bornivorous, E is said to be *b-Baire-like*, cf. [17]. Clearly Baire ⇒ Baire-like ⇒ barrelled. Every metrizable lcs is b-Baire-like. Recall that a lcs E is *barrelled*, if every closed absolutely convex and absorbing subset of E is a neighbourhood of zero of E. Every metrizable barrelled space is Baire-like. See [11–13,17] for another conditions for a lcs to be Baire-like. The Baire-likeness of some concrete normed vector-valued function spaces were studied by several specialists, see [8] for details. We note only that the normed spaces of Pettis or Bochner integrable functions are not Baire spaces but Baire-like, cf. [6,7]. There is an interesting connection of barrelled and Baire-like spaces with closed graph theorems, cf. [16,18]. For instance, Saxon proved that Grothendieck's factorization theorem for closed linear maps from a Baire lcs into an (*LF*)-space remains true for closed linear maps from a Baire-like space into an (*LB*)-space. In contrast to the Baire spaces, Baire-like spaces enjoy good permanence properties.

An increasing sequence (A_n) of absolutely convex and closed subsets of a lcs E is *absorbing* (respectively *bornivorous*) if it covers E (respectively if for every bounded subset B of E there exists $n \in \mathbb{N}$ such that $B \subset A_n$). In a barrelled space every absorbing sequence is bornivorous, cf. [16, 8.1.23]. Recall that if D is an absolutely convex subset of $C_c(X,E)$, a *hold* K of D is a compact subset of βX such that $f \in C(X,E)$ belongs to D if its continuous extension f^β of βX into βE is identically zero on a neighbourhood of K. The intersection $k(D)$ of all holds of an absolutely convex set D in $C_c(X,E)$ is again a hold [19, II.1.2], and it is called a *support* of D. If moreover D is bornivorous, then $k(D)$ is contained in υX [19, II.2.4, II.1.2, II.1.4].

In this part we apply Theorem 1 to show the following result which improves results of [19, IV.4] and [11–13].

Theorem 2. *Let X be a strongly Hewitt space and E a metrizable lcs. Then $C_c(X,E)$ is b-Baire-like, and $C_c(X,E)$ is Baire-like iff E is barrelled. Hence, if X is locally compact and E is metrizable, $C_c(X,E)$ is Baire-like and bornological iff E is barrelled and X is Hewitt.*

Proof. Let $\mathcal{F}(E)$ denotes the set of continuous seminorms defining the topology of E. Let (D_n) be a bornivorous sequence in $C_c(X,E)$. First we prove that $k(D_{m_0}) \subset X$ for some $m_0 \in \mathbb{N}$. If this fails, for every $n \in \mathbb{N}$ there exists $x_n \in k(D_n) \setminus X$. Let $f : \beta X \to [0,1]$ be a continuous function which is positive on X and vanishes on some subsequence of (x_n). Since (D_n) is increasing we may assume that $f(x_n) = 0,\ n \in \mathbb{N}$. The sets $A_m = \{y \in \beta X\colon\ f(y) > m^{-1}\}$ are open in βX and compose an increasing sequence which covers X. Since $x_n \notin \overline{A_n}$ for $n \in \mathbb{N}$, where the closure is taken in βX,

$$k(D_n) \not\subset \overline{A_n}$$

for every $n \in \mathbb{N}$. This implies that $\overline{A_n}$ is not a hold of D_n for any $n \in \mathbb{N}$. Hence there exists a sequence $f_n \in C_c(X,E) \setminus D_n$ such that its extension $f_n^\beta = 0$ on some neighbourhood of $\overline{A_n}$. Since (f_n) converges to zero in $C_c(X,E)$, there exists $p \in \mathbb{N}$ such that $f_n \in D_p$ for all $n \in \mathbb{N}$, a contradiction. We proved that there exists $m_0 \in \mathbb{N}$ such that $k(D_{m_0}) \subset X$.

Let (p_n) be an increasing sequence in $\mathcal{F}(E)$ giving the topology of E. We prove that (+) there exist $m \geqslant m_0$, $r > 0$ and a continuous seminorm p on E such that $\{f \in C(X,E)\colon\ \sup_{x \in k(D_m)} p(f(x)) < r\} \subset D_m$. Since $(k(D_n))_{n \geqslant m_0}$ is decreasing, (+) will imply that $C_c(X, E)$ is b-Baire-like. But in order to get $(+)$ it is enough to show, cf. [19, II.1.4], that $(*)$ there exist $p \in \mathcal{F}(E)$ and $r > 0$, $n \geqslant m_0$, such that

$$\Bigl\{f \in Y\colon \sup_{x\in X} p\bigl(f(x)\bigr) < r\Bigr\} \subset D_n.$$

Assume that $(*)$ fails. Then there exists a sequence (f_n) in Y, $f_n \in Y \setminus D_n$, $p_n(f_n(x)) < n^{-1}$ for every $x \in X$. Since (f_n) converges to zero in $C_c(X, E)$ and (D_n) is bornivorous we get a contradiction.

Now assume E is barrelled. By the Nachbin's theorem, cf. [16, 10.1.20], the space $C_c(X)$ is barrelled and applying Mendoza's theorem, cf. [19, IV.7.9], one gets that $C_c(X, E)$ is barrelled. Since in a barrelled space any absorbing sequence is bornivorous, the first conclusion of Theorem 2 applies to deduce that $C_c(X, E)$ is Baire-like. Finally assume that X is locally compact and E is metrizable, $C_c(X, E)$ is Baire-like and bornological. Then E is barrelled as complemented in $C_c(X, E)$. By [19, IV.4.2], X is Hewitt. If X is Hewitt and E is metrizable, $C_c(X, E)$ is bornological [19, IV.4.2]. The rest follows from the first part of Theorem 2. □

Corollary 2. *Let X be a space of pointwise countable type. Then $C_c(X)$ is bornological and Baire-like iff X is strongly Hewitt.*

Proof. Assume $C_c(X)$ is bornological and Baire-like. Then X is Hewitt. We prove that X is locally compact. Let $x \in X$. Since X is of pointwise countable type, there exist a compact set K in X containing x and a decreasing basis (U_n) of open neighbourhoods of the set K. Then the absolutely convex and closed sets $W_n = \{f \in C(X)\colon \sup_{x\in U_n} |f(x)| \leqslant n\}$ cover $C_c(X)$. By assumption there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ and a compact subset S of X such that

$$\Bigl\{f \in C(X)\colon \sup_{y\in S}\bigl|f(y)\bigr| < \varepsilon\Bigr\} \subset W_n.$$

Hence $U_n \subset S$. We proved that X is locally compact and Theorem 1 applies to deduce that X is strongly Hewitt. For the converse we apply Theorems 1 and 2. □

From Theorem 2 it follows that for any strongly Hewitt space X the space $C_c(X)$ is Baire-like. Nevertheless, one gets the following

Example C. *There exists a locally compact and strongly Hewitt space X such that the space $C_c(X)$ is not Baire*. Consider the locally compact and Hewitt space X from Example 1.11 of [4]; recall that X is the set $\mathbb{R}$ endowed with a topology defined as follows:

(a) For every $t \in \mathbb{Q}$ the set $\{t\}$ is open in X.

(b) For every $t \in \mathbb{R} \setminus \mathbb{Q}$ there exists a sequence (t_n) in $\mathbb{Q}$ which converges to t such that the sets $V_n(t) = \{t\} \cup \{t_m\colon m \geqslant n\}$, $n, m \in \mathbb{N}$, form a base of neighbourhoods of t in X.

(c) For all dense sets $A, B \subset \mathbb{Q}$ in the natural topology of $\mathbb{R}$ the set $\overline{A} \cap \overline{B}$ is non-empty, where the closure is taken in X.

To show that $C_c(X)$ is not Baire it is enough to prove that there exists an infinite family $\mathcal{K}$ of non-empty compact subsets of X such that:

(o) For every compact set L of X there exists $K \in \mathcal{K}$ with $K \cap L = \emptyset$,

(oo) any infinite subfamily of $\mathcal{K}$ is not discrete, cf. [14, 5.3.5].

Let (P_n) be a sequence of pairwise disjoint finite subsets of $\mathbb{R} \setminus \mathbb{Q}$ such that for any subsequence (P_{n_k}) the set $\bigcup_k P_{n_k}$ is dense in $\mathbb{R}$. Let $P_n = \{t_k^n\colon 1 \leqslant k \leqslant m_n\}$ for all $n \in \mathbb{N}$. For all $n, m \in \mathbb{N}$ the set

$$K_{n,m} = \bigcup_{1 \leqslant k \leqslant m_n} V_m\bigl(t_k^n\bigr)$$

is non-empty and compact in X. Observe that the family $\mathcal{K} = \{K_{n,m}\colon n, m \in \mathbb{N}\}$ satisfies (o). Indeed, any compact subset L of X is contained in a set of the form

$$V_1\bigl(t^1\bigr) \cup \cdots \cup V_1\bigl(t^m\bigr) \cup \{p_1, \ldots, p_k\},$$

where $m, k \in \mathbb{N}$, $t^1, t^2, \ldots, t^m \in \mathbb{R} \setminus \mathbb{Q}$, $p_1, \ldots, p_k \in \mathbb{R}$. Indeed, the set $L \cap (\mathbb{R} \setminus \mathbb{Q})$ is finite, since the open cover $\{V_1(t)\colon t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\} \cup \{\{q\}\colon q \in L \cap \mathbb{Q}\}$ of L has a finite subcover. The set $(L \setminus (\bigcup\{V_1(t)\colon t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\})$ is finite, since $X^d \subset (\mathbb{R} \setminus \mathbb{Q})$. This implies that $\mathcal{K}$ satisfies (o). Finally we prove that $\mathcal{K}$ satisfies (oo), too. Assume for the converse that some infinite subfamily $\{K^n\colon n \in \mathbb{N}\}$ of $\mathcal{K}$ is discrete. Then the sets $A = \bigcup_n K^{2n}$, $B = \bigcup_n K^{2n+1}$ are disjoint and closed in X. Using the property of the sequence (P_n) we derive that $A \cap \mathbb{Q}$ and $B \cap \mathbb{Q}$ are dense in $\mathbb{R}$. Hence, by (c), the set $\overline{(A \cap \mathbb{Q})} \cap \overline{(B \cap \mathbb{Q})}$ is non-empty, a contradiction. We proved that $C_c(X)$ is not Baire.

Recall, that Nagata's fundamental theorem states that if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic, then the spaces X and Y are homeomorphic, cf. [1]. This suggests the following problem, cf. [1]: *Two spaces X and Y are called ℓ-equivalent if the corresponding* (locally convex) *spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. What topological invariants are preserved under the ℓ-equivalence*?

It is known, cf. [1], that many of topological properties like local compactness, metrizability, the Fréchet–Urysohn property, the weight, the character, etc., are not preserved under ℓ-equivalence. For "positive" results see [3]. We note only that if X and Y are ℓ-equivalent, then X is a Hewitt space iff Y is a Hewitt space cf. [19]. This motivates the following

Theorem 3. *Let X be strongly Hewitt and Y of pointwise countable type. If X and Y are ℓ-equivalent, then Y is strongly Hewitt.*

Proof. Since the Hewitt property is inherited by the ℓ-equivalence, Y is Hewitt. Hence X and Y are μ-spaces and consequently $C_c(X)$, $C_c(Y)$ are linearly homeomorphic. Now the conclusion follows from Corollary 2. □

Acknowledgements

The authors wish to thank Professor M. Henriksen for providing the reference [4], Professors A.V. Arhangel'skii, R.A. McCoy and A. Todd for their remarks concerning the subject developed in this note.

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